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# Canonical Miura maps between the modified KP and KP hierarchies

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**Abstract.** We investigate the Hamiltonian nature of two Miura maps between the modified KP and KP hierarchies. We show that they are canonical, in the sense that the bi-Hamiltonian structure of the modified KP hierarchy is mapped to the bi-Hamiltonian structure of the KP hierarchy.

## 1. Introduction

The Miura map [1] has been playing an important role in the development of soliton theory [2]. It is a transformation between two nonlinear equations, which in general cannot be solved easily. However, knowing the solutions of one of the nonlinear systems, one may obtain the solutions of the other one via an appropriate Miura map. A typical example is the Miura map between the KdV equation and the modified KdV equation. It is not only the key step to prove the existence of an infinite number of conservation laws for the KdV equation, but also provides the starting point for the inverse scattering method [3]. Therefore, finding the Miura maps between different integrable systems is quite useful in order to obtain their solutions. Moreover, since almost all the known integrable systems are Hamiltonian, exploring the Hamiltonian nature of these Miura maps will deepen our understanding of the relationship between these integrable systems.

Recently, Kupershmidt [4] investigated the canonical property of the Miura map between the modified KP (mKP) hierarchy and the KP hierarchy. He showed that the Miura map (we call it  $G_1$  in equation (23)) [5, 6] is a canonical map in the sense that the first Hamiltonian structure of the mKP hierarchy is mapped to the first Hamiltonian structure of the KP hierarchy. The same conclusion was also reached by using the  $R$ -matrix approach [7, 8]. However, due to the fact that the bi-Hamiltonian structure is one of the most important properties of those hierarchies, one would like to know whether the Miura map  $G_1$  can be a canonical map with respect to the *second* Hamiltonian structure as well. Furthermore, it has been shown [9] that there exists another Miura map (we call it  $G_2$  in (24)), which together with  $G_1$ , enables us to construct the Bäcklund transformations to the mKP and KP hierarchies themselves [9]. Therefore, it is also quite interesting to investigate the Hamiltonian nature of this Miura map.

In this paper we will concentrate on the canonical property of the Miura maps between the mKP and KP hierarchies. We first show that two transformations  $G_1$  and  $G_2$  ((23) and

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(24)) define Miura maps between the mKP and KP hierarchies. Then we prove that these two Miura maps are all canonical, in the sense that the *bi-Hamiltonian* structure of the mKP hierarchy is mapped to the *bi-Hamiltonian* structure of the KP hierarchy.

Our paper is organized as follows: section 2 contains definitions and statements of our main results. The proofs are given in section 3 and section 4 is the conclusion.

## 2. Definitions and main results

The pseudo-differential operator is defined by

$$\Lambda = \sum_{n=0}^N a_n \partial^n + \sum_{n<0} \partial^n a_n \quad (1)$$

with coefficients  $a_n$  depending on an infinite set of variables  $t_1 \equiv x, t_2, t_3, \dots$ . For positive  $n$  the symbol  $\partial^n$  is the power of the differential operator  $\partial = \partial/\partial x$ , whereas negative powers are formal integrations defined by the generalized Leibniz rule

$$\partial^{-n} f = \sum_{l=0}^{\infty} (-1)^l \binom{n+l-1}{l} f^{(l)} \partial^{-n-l} \quad (n > 0). \quad (2)$$

The projections of (1) to various differential operators are denoted by

$$(\Lambda)_n = \begin{cases} a_n \partial^n & \text{for } n \geq 0 \\ \partial^n a_n & \text{for } n < 0 \end{cases} \quad (3)$$

and using the notations  $\Lambda_+ = (\Lambda)_{\geq 0}$  and  $\Lambda_- = (\Lambda)_{< 0}$  for short.

The residue of the pseudo-differential operators is defined by

$$\text{res } \Lambda = a_{-1} \quad (4)$$

which gives rise to a trace formalism on  $\Lambda$ ,

$$\text{Tr } \Lambda = \int \text{res } \Lambda \quad (5)$$

such that for any two pseudo-differential operators  $A$  and  $B$ ,  $\text{Tr}([A, B]) = 0$ . Therefore, one can define a symmetric duality bracket

$$\langle A, B \rangle = \text{Tr}(AB) = \langle B, A \rangle \quad (6)$$

in the space of the pseudo-differential operator.

In the following, we list some useful identities [10] which simplify the computations involving compositions of pseudo-differential operators

$$(\Lambda^*)_+ = (\Lambda_+)^* \quad (\Lambda^*)_- = (\Lambda_-)^* \quad (7)$$

$$\text{res}(\Lambda) = -\text{res}(\Lambda^*) \quad \text{res}(\Lambda)_x = \text{res}(\partial \Lambda - \Lambda \partial) \quad (8)$$

$$\text{res}(\Lambda \partial^{-1}) = (\Lambda)_0 \quad \text{res}(\partial^{-1} \Lambda) = (\Lambda^*)_0 \quad (9)$$

$$(\Lambda \partial^{-1})_- = (\Lambda)_0 \partial^{-1} + (\Lambda)_- \partial^{-1} \quad (10)$$

$$(\partial^{-1} \Lambda)_- = \partial^{-1} (\Lambda^*)_0 + \partial^{-1} (\Lambda)_- \quad (11)$$

where  $*$  stands for the conjugate operation:  $(AB)^* = B^* A^*$ ,  $\partial^* = -\partial$ ,  $f(x)^* = f(x)$ .

The KP hierarchy [2, 11] is defined by the pseudo-differential operator  $L$  of the form

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + u_3 \partial^{-3} + \dots \quad (12)$$

and imposing the evolution equations,

$$\frac{dL}{dt_n} = [L_+^n, L] \quad n = 1, 2, 3, \dots \tag{13}$$

$$\frac{\partial \phi}{\partial t_n} = (L_+^n \phi)_0 \tag{14}$$

$$\frac{\partial \bar{\phi}}{\partial t_n} = -((L_+^n)^* \bar{\phi})_0 \tag{15}$$

where  $\phi$  and  $\bar{\phi}$  are called the eigenfunction and adjoint eigenfunction, respectively.

It is well known that the compatible bi-Hamiltonian structure of the KP hierarchy (see, for example, [2]) is given by

$$\Omega_1: \frac{\delta H}{\delta L} \rightarrow \left[ \left( \frac{\delta H}{\delta L} \right)_+, L \right] - \left[ \frac{\delta H}{\delta L}, L \right]_+ \tag{16}$$

$$\Omega_2: \frac{\delta H}{\delta L} \rightarrow \left( L \frac{\delta H}{\delta L} \right)_+ L - L \left( \frac{\delta H}{\delta L} L \right)_+ + \left[ L, \int^x \text{res} \left( \left[ L, \frac{\delta H}{\delta L} \right] \right) \right] \tag{17}$$

with

$$\frac{\delta H}{\delta L} = \frac{\delta H}{\delta u_1} + \partial \frac{\delta H}{\delta u_2} + \partial^2 \frac{\delta H}{\delta u_3} + \dots \tag{18}$$

such that the hierarchy equations (13) have a Hamiltonian description.  $\Omega_1$  is given by Adler–Kostant–Symes construction [12] and  $\Omega_2$  is just the Gelfand–Dickey’s quadratic structure [13] with a Dirac constraint imposed by  $u_0 = 0$  in (12).

Another integrable system which is intimately related to the KP hierarchy is the mKP hierarchy [5–7] defined by

$$\frac{dK}{dt_n} = [(K^n)_{\geq 1}, K] \quad n = 1, 2, 3, \dots \tag{19}$$

where

$$K = \partial + v_0 + v_1 \partial^{-1} + v_2 \partial^{-2} + v_3 \partial^{-3} + \dots \tag{20}$$

The compatible bi-Hamiltonian structure for the mKP hierarchy is given [14, 15] by

$$\Theta_1: \frac{\delta H}{\delta K} \rightarrow \left[ \left( \frac{\delta H}{\delta K} \right)_{\geq 1}, K \right] - \left( \left[ \frac{\delta H}{\delta K}, K \right] \right)_{\geq -1} \tag{21}$$

$$\Theta_2: \frac{\delta H}{\delta K} \rightarrow \left( K \frac{\delta H}{\delta K} \right)_+ K - K \left( \frac{\delta H}{\delta K} K \right)_+ - \left[ \left( K \frac{\delta H}{\delta K} \right)_0, K \right] - \left( \left[ \frac{\delta H}{\delta K}, K \right] \right)_{-1} K + \left[ \int^x \text{res} \left( \left[ \frac{\delta H}{\delta K}, K \right] \right), K \right]. \tag{22}$$

Let us consider the following two maps:

$$G_1: K \rightarrow L_1 = z_0^{-1} K z_0 \tag{23}$$

$$G_2: K \rightarrow L_2 = z_0^{-1} \partial K \partial^{-1} z_0 \tag{24}$$

where  $K$  is defined by (20) and  $z_0 = \exp(-\int^x v_0)$ .

It is straightforward to see that both  $L_1$  and  $L_2$  preserve the form of the Lax operator in (12). Using the generalized Leibniz rule in (12), the coefficients  $u_i^{(k)}$  in  $L_k$  ( $k = 1, 2$ )

and  $v_i$  in  $K$  have the following correspondences

$$\begin{aligned}
 u_m^{(1)} &= \sum_{i=1}^m (-1)^{m-i} \binom{m-1}{m-i} z_0^{-1} z_0^{(m-i)} v_i && \text{for } G_1 \\
 u_m^{(2)} &= \sum_{i=1}^m (-1)^{m-i} \binom{m-1}{m-i} z_0^{-1} z_0^{(m-i)} (v'_{i-1} + v_i) && \text{for } G_2
 \end{aligned}
 \tag{25}$$

where  $m \geq 1$ .

Now we want to verify that both  $L_1$  and  $L_2$  satisfy the hierarchy equations (13), if  $K$  is a solution of (19).

*Lemma 2.1.*  $z_0^{-1}$  satisfies (14) with  $L = L_1$  defined by (23), whereas  $z_0$  satisfies (15) with  $L = L_2$  defined by (24).

*Proposition 2.2.* If  $K$  satisfies the Lax equation (19), then the transformed operators  $L_1$  and  $L_2$  satisfy (13).

Thus, we know that  $z_0^{-1}$  is an eigenfunction of  $L_1$  and  $z_0$  is an adjoint eigenfunction of  $L_2$ , respectively, and the transformations  $G_1$  and  $G_2$  define two Miura maps between the mKP and KP hierarchies. In fact, these two particular Miura maps  $G_1$  and  $G_2$  have been used [9] to obtain the Bäcklund transformations for the KP hierarchy [10, 16, 17] and the mKP hierarchy [6, 8, 9].

As we have seen above, both the KP hierarchy and the mKP hierarchy equip a compatible bi-Hamiltonian structure. So it is quite natural to ask whether the bi-Hamiltonian structures are still preserved under the Miura maps  $G_1$  and  $G_2$ . To investigate the canonical property of a Miura map, we need the linearized map and its transposed map.

*Lemma 2.3.* For the Miura map  $G_1$ , the linearized map  $G'_1$  and its transposed map  $G_1^{\dagger}$  are given by

$$G'_1: B \rightarrow z_0^{-1} B z_0 + \left[ \int^x b_0, L_1 \right] \tag{26}$$

$$G_1^{\dagger}: A \rightarrow z_0 A z_0^{-1} + \partial^{-1} \left( \int^x \text{res}[A, L_1] \right). \tag{27}$$

On the other hand, for the Miura map  $G_2$ , the linearized map  $G'_2$  and its transposed map  $G_2^{\dagger}$  are given by

$$G'_2: B \rightarrow z_0^{-1} \partial B \partial^{-1} z_0 + \left[ \int^x b_0, L_2 \right] \tag{28}$$

$$G_2^{\dagger}: A \rightarrow \partial^{-1} z_0 A z_0^{-1} \partial + \partial^{-1} \left( \int^x \text{res}[A, L_2] \right) \tag{29}$$

where  $A$  and  $B$  are any pseudo-differential operators,  $b_0 \equiv (B)_0$  and  $\dagger$  is the transposed operation defined by  $\langle A, G'B \rangle = \langle G'^{\dagger} A, B \rangle$ .

We are now in a position to investigate the canonical property of the Miura maps  $G_1$  and  $G_2$ .

*Proposition 2.4.* The Miura maps  $G_1$  and  $G_2$  map the bi-Hamiltonian structure of the mKP hierarchy given by  $\Theta_1$  and  $\Theta_2$  to the bi-Hamiltonian structure of the KP hierarchy given by  $\Omega_1$  and  $\Omega_2$  respectively, i.e.

$$\Omega_1 = G'_1 \Theta_1 G_1{}^\dagger \tag{30}$$

$$\Omega_2 = G'_1 \Theta_2 G_1{}^\dagger \tag{31}$$

$$\Omega_1 = G'_2 \Theta_1 G_2{}^\dagger \tag{32}$$

$$\Omega_2 = G'_2 \Theta_2 G_2{}^\dagger \tag{33}$$

where  $G'_i$  and  $G_i{}^\dagger$  are defined in lemma 2.3.

Therefore both of the Miura maps,  $G_1$  and  $G_2$  are canonical maps. This is our main result.

### 3. The proofs

*Proof of lemma 2.1.* Taking the zeroth order terms on both sides of (19), we obtain

$$\frac{dv_0}{dt_n} = [K_{\geq 1}^n, K]_0 = (K^n)_{0x}. \tag{34}$$

For the map  $G_1$ , the above equation becomes

$$\frac{dv_0}{dt_n} = (z_0((L_1)_+^n z_0^{-1})_0)_x \tag{35}$$

which implies

$$\frac{dz_0^{-1}}{dt_n} = ((L_1)_+^n z_0^{-1})_0. \tag{36}$$

On the other hand, for the map  $G_2$ , we have

$$\frac{dv_0}{dt_n} = (\partial^{-1} z_0 (L_2)_+^n z_0^{-1} \partial)_{0x} = \text{res}(\partial^{-1} z_0 (L_2)_+^n z_0^{-1})_x = (z_0^{-1} ((L_2)_+^{n*} z_0)_0)_x \tag{37}$$

which implies

$$\frac{dz_0}{dt_n} = -((L_2)_+^{n*} z_0)_0. \tag{38}$$

This completes the proof of lemma 2.1. □

*Proof of proposition 2.2.* From (23) and (24), we have

$$\begin{aligned} \frac{dL_1}{dt_n} &= \frac{dz_0^{-1}}{dt_n} K z_0 + z_0^{-1} \frac{dK}{dt_n} z_0 + z_0^{-1} K \frac{dz_0^{-1}}{dt_n} = [z_0((L_1)_+^n z_0^{-1})_0, L_1] + [(L_1)_+^n \\ &\quad - z_0^{-1} (z_0 (L_1)_+^n z_0^{-1})_{\leq 0} z_0, L_1] = [(L_1)_+^n, L_1]. \end{aligned} \tag{39}$$

On the other hand, from (24) and (38), we have

$$\begin{aligned} \frac{dL_2}{dt_n} &= \frac{dz_0^{-1}}{dt_n} \partial K \partial^{-1} z_0 + z_0^{-1} \partial \frac{dK}{dt_n} \partial^{-1} z_0 + z_0^{-1} \partial K \partial^{-1} \frac{dz_0^{-1}}{dt_n} \\ &= [z_0^{-1} ((L_2)_+^{n*} z_0)_0, L_2] + [(L_2)_+^n, L_2] - [z_0^{-1} \partial (\partial^{-1} z_0 (L_2)_+^n z_0^{-1} \partial)_{\leq 0} \partial^{-1} z_0, L_2]. \end{aligned} \tag{40}$$

The first entity in the last commutator of (40) can be simplified as follows

$$\begin{aligned}
 z_0^{-1} \partial (\partial^{-1} z_0 (L_2)_+^n z_0^{-1} \partial)_{\leq 0} \partial^{-1} z_0 &= z_0^{-1} \partial \{ (\partial^{-1} z_0 (L_2)_+^n z_0^{-1} \partial)_{< 0} + (\partial^{-1} z_0 (L_2)_+^n z_0^{-1} \partial)_0 \} \partial^{-1} z_0 \\
 &= z_0^{-1} \partial \{ -\partial^{-1} (\partial z_0^{-1} (L_2)_+^{n*} z_0)_0 + \text{res}(\partial^{-1} z_0 (L_2)_+^n z_0^{-1}) \} \partial^{-1} z_0 \\
 &= -z_0^{-1} (z_0^{-1} (L_2)_+^{n*} z_0)_{0x} \partial^{-1} z_0 + z_0^{-1} \partial (z_0^{-1} (L_2)_+^n z_0^{-1})_0 \partial^{-1} z_0 \\
 &= z_0^{-1} ((L_2)_+^{n*} z_0)_0.
 \end{aligned} \tag{41}$$

Therefore, the first and the third commutators in (40) cancel each other, and (13) is obtained. This completes the proof of proposition 2.2.  $\square$

*Proof of lemma 2.3.* To prove (26), consider an infinitesimal deformation (say,  $B$ ) of the operator  $K$ . Then the map  $G_1$  becomes

$$\begin{aligned}
 K + B &\rightarrow \exp \left( \int^x (v_0 + b_0) \right) (K + B) \exp \left( - \int^x (v_0 + b_0) \right) \\
 &= z_0^{-1} K z_0 + z_0^{-1} B z_0 + \left[ \int^x b_0, L_1 \right] + O(B^2)
 \end{aligned} \tag{42}$$

and the corresponding linearized map  $G'_1$  is given by

$$G'_1: B \rightarrow z_0^{-1} B z_0 + \left[ \int^x b_0, L_1 \right]. \tag{43}$$

Moreover, using the definitions (5) and (6), we have

$$\begin{aligned}
 \langle A, G'_1 B \rangle &= \int \text{res}(A G'_1 B) = \int \text{res}(A z_0^{-1} B z_0) + \int \text{res} \left( A \left[ \int^x b_0, L_1 \right] \right) \\
 &= \int \text{res}(z_0 A z_0^{-1} B) + \int b_0 \int^x \text{res}[A, L_1] \\
 &= \int \text{res}(z_0 A z_0^{-1} B) + \int \text{res} \left( \partial^{-1} \left( \int^x \text{res}[A, L_1] \right) B \right) = \langle G_1^\dagger A, B \rangle
 \end{aligned} \tag{44}$$

here we have used integration by part to reach the third line and  $b_0 = \text{res}(B \partial^{-1})$  to reach the fourth line. Comparing the last two lines of (44), the transposed map  $G_1^\dagger$  can be read as

$$G_1^\dagger: A \rightarrow z_0 A z_0^{-1} + \partial^{-1} \int^x \text{res}[A, L_1]. \tag{45}$$

For the proofs of (28) and (29), the procedures are similar to those for  $G_1$ , therefore we skip them here.  $\square$

*Proof of proposition 2.4.* To prove (30) and (31) for  $G_1$ , let us act on the right-hand side of (30) on an arbitrary pseudo-differential operator  $A$ , then

$$G'_1 \Theta_1 G_1^\dagger A = G'_1 B \tag{46}$$

where

$$\begin{aligned}
 B &\equiv \Theta_1 G_1^\dagger A = [(z_0 A z_0^{-1})_{\geq 1}, K] - [(z_0 A z_0^{-1}), K]_{\geq -1} + \partial^{-1} \text{res}[A, L_1] \\
 &= [(z_0 A z_0^{-1}), K]_- - [(z_0 A z_0^{-1})_{\leq 0}, K]
 \end{aligned} \tag{47}$$

and then

$$\int^x b_0 = \int^x (B)_0 = (z_0 A z_0^{-1})_0. \tag{48}$$

Substituting (47) and (26) into (26), we obtain

$$G'_1 \Theta_1 G_1^\dagger A = [A_+, L_1] - [A, L_1]_+ = \Omega_1 A. \tag{49}$$

To prove (31) for the second Hamiltonian structure, we use (22) and (27), and obtain

$$B = \Theta_2 G_1^\dagger A = (K G_1^\dagger A)_+ K - K (G_1^\dagger A K)_+ - [(K G_1^\dagger A)_0, K] - ([G_1^\dagger A, K])_{-1} K + \left[ \int^x \text{res}([G_1^\dagger A, K]), K \right] \tag{50}$$

where each term in (50) can be calculated as follows:

$$(1) = z_0(L_1 A)_+ L_1 z_0^{-1} + \left( \int^x \text{res}[A, L_1] \right) K \tag{51}$$

$$(2) = -z_0 L_1 (A L_1)_+ z_0^{-1} - K \int^x \text{res}[A, L_1] \tag{52}$$

$$(3) = -[(z_0 L_1 A z_0^{-1})_0 + \int^x \text{res}[A, L_1], K] \tag{53}$$

$$(4) = (5) = 0. \tag{54}$$

Then

$$B = (1) + (2) + (3) + (4) + (5) = z_0(L_1 A)_+ L_1 z_0^{-1} - z_0 L_1 (A L_1)_+ z_0^{-1} - [(z_0 L_1 A z_0^{-1})_0, K] \tag{55}$$

and

$$\int^x b_0 = (z_0 L_1 A z_0^{-1})_0 + \int^x \text{res}[A, L_1]. \tag{56}$$

Substituting (55) and (56) into (26), we obtain

$$G'_1 \Theta_2 G_1^\dagger A = (L_1 A)_+ L_1 - L_1 (A L_1)_+ + \left[ L_1, \int^x \text{res}[L_1, A] \right] = \Omega_2 A. \tag{57}$$

For  $G_2$ , (32) and (33) can also be proved in a similar manner. Therefore, for the first Hamiltonian structure, we have

$$B = \Theta_1 G_2^\dagger A = \partial^{-1} z_0 \{ [A_+, L_2] - [A, L_2]_+ - [z_0^{-1} (A^* z_0)_0, L_2] \} z_0^{-1} \partial \tag{58}$$

and

$$\int^x b_0 = z_0^{-1} (A^* z_0)_0. \tag{59}$$

Substituting (58) and (59) into (28), we have

$$G'_2 \Theta_1 G_2^\dagger A = [A_+, L_2] - [A, L_2]_+ = \Omega_1 A. \tag{60}$$

Finally, to prove (33) for the second Hamiltonian structure, using (22) and (29), we have

$$B = \Theta_2 G_2^\dagger A = (K G_2^\dagger A)_+ K - K (G_2^\dagger A K)_+ - [(K G_2^\dagger A)_0, K] - ([G_2^\dagger A, K])_{-1} K + \left[ \int^x \text{res}([G_2^\dagger A, K]), K \right] \tag{61}$$



where each term in (61) can be calculated as follows

$$(1) = \partial^{-1} z_0 (L_2 A)_+ L_2 z_0^{-1} \partial + \partial^{-1} (z_0^{-1} ((L_2 A)_+^* z_0)_0)_x K + \left( \int^x \text{res}[A, L_2] \right) K \quad (62)$$

$$(2) = -\partial^{-1} z_0 L_2 (A L_2)_+ z_0^{-1} \partial - K \partial^{-1} (z_0^{-1} ((A L_2)_+^* z_0)_0)_x - K \left( \int^x \text{res}[A, L_2] \right) \quad (63)$$

$$(3) = - \left[ z_0^{-1} ((L_2 A)_+^* z_0)_0 + \int^x \text{res}[A, L_2], K \right] \quad (64)$$

$$(4) = \partial^{-1} (z_0^{-1} ([A, L_2]_+^* z_0)_0)_x K \quad (65)$$

$$(5) = -[z_0^{-1} ([A, L_2]_+^* z_0)_0, K]. \quad (66)$$

Then

$$B = (1) + (2) + (3) + (4) + (5) = \partial^{-1} z_0 \{ (L_2 A)_+ L_2 - L_2 (A L_2)_+ - [z_0^{-1} ((A L_2)_+^* z_0)_0, L_2] \} z_0^{-1} \partial \quad (67)$$

and

$$\int^x b_0 = z_0^{-1} ((A L_2)_+^* z_0)_0 + \int^x \text{res}[A, L_2]. \quad (68)$$

Substituting (67) and (68) into (28), we have

$$G_2' \Theta_2 G_2^{\dagger} A = (L_2 A)_+ L_2 - L_2 (A L_2)_+ + \left[ L_2, \int^x \text{res}[L_2, A] \right] = \Omega_2 A. \quad (69)$$

This finishes the proof of proposition 2.4.  $\square$

#### 4. Conclusions

The canonical property of the Miura map  $G_1$  between the mKP and KP hierarchies was first conjectured [18] and then proved [4] by Kupershmidt for the first Hamiltonian structure. Based on this observation, we extend his result to the second Hamiltonian structure. Moreover, we have shown that there is still another less studied Miura map  $G_2$  which also possess the canonical property. Therefore, the results presented here provide a deeper understanding of the Hamiltonian nature of the Miura maps,  $G_1$  and  $G_2$ . Our approach only involves the algebra of the pseudo-differential operators, hence the proofs are simpler and more straightforward. Further extensions and generalizations along the same line are possible and interesting. For example, one may investigate the canonical property of the Miura maps which relate the KP hierarchy to other non-standard integrable hierarchies [6, 8], or the KP hierarchy with constraints [15]. We expect that the Miura maps between these integrable hierarchies should also be canonical. We will leave these discussions to another publication.

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